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Restriction of a cuspidal module for finite general linear groups

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Abstract

In this article, we consider two subgroups A and A' of $\mathrm{GL}_n(q)$ containing a fixed unipotent subgroup U, either of which may be identified with the group of all affine transformations of an $n-1$ dimensional affine space. We describe the restriction of an irreducible cuspidal module to the subgroup $A \cap A'$ of $\mathrm{GL}_n(q)$ and prove that it is multiplicity-free.

1. Finite general linear groups

Let $G_n = \mathrm{GL}_n(q)$, the general linear group of degree n over a finite field $\mathbb{F}_q$. Let B_n be the group of upper triangular matrices (a Borel subgroup) and H_n the diagonal matrices. For each $i \in \{1, 2, \ldots, n-1\}$, define s_i be the permutation matrix corresponding to the transposition $(i\ i+1)$. Let $S = \{s_i \mid 1 \leqslant i \leqslant n-1\}$. Then $W_n = \langle S \rangle$ is the Weyl group and isomorphic to the symmetric group on $\{1, 2, \ldots, n\}$. We shall frequently identify elements of W_n with the corresponding permutations.

Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis for n-dimensional Euclidean space and let W act on this space by permuting the e_i. Write $\alpha_{ij} = e_i - e_j$. Then the set

$$\Phi = \{\alpha_{ij} \mid i \neq j,\ 1 \leqslant i, j \leqslant n\}.$$

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can be identified with the root system of W. Moreover, $\Phi^+ = \{\alpha_{ij} \in \Phi \mid i < j\}$ (the positive roots), $\Phi^- = \{\alpha_{ij} \in \Phi \mid i > j\}$ (the negative roots), $\Pi = \{\alpha_{ij} \in \Phi \mid j = i + 1\}$ (the simple roots).

For each $J \subseteq S$, let $W_J := \langle J \rangle$. Then (W_J, J) is a Coxeter system with simple roots $\Pi_J = \{\alpha_s \mid s \in J\}$ and root system $\Phi_J = W_J(\Pi_J)$.

The subgroup $U_n = \{(a_{ij}) \in B_n \mid a_{ii} = 1\}$ (upper unitriangular matrices) is a maximal unipotent subgroup and Sylow p-subgroup of G_n. We have $B_n = U_n \rtimes H_n$. For each $\alpha = \alpha_{ij} \in \Phi$, define

$$x_\alpha(t) = x_{ij}(t) = I_n + t\mathbf{e}_{ij}$$

for any $t \in \mathbb{F}_q$, where I_n is the identity matrix. The group $X_\alpha = X_{ij} = \{x_{ij}(t) \mid t \in \mathbb{F}_q\}$ is the root subgroup corresponding to α. For any $w \in W$,

$$w x_{ij}(t) w^{-1} = x_{w(i),w(j)}(t). \tag{1}$$

Furthermore, $U_n = \prod_{1 \leqslant i < j \leqslant n} X_{ij}$.

2. Main result

From now on, assume that $k, n \in \mathbb{Z}^+$ with $0 \leqslant k \leqslant n$, $G_k = \mathrm{GL}_k(q)$ and $G = G_n$.

For each k, define

$$\begin{aligned}
A_k &= \bigl\{(a_{ij}) \in G_k \mid a_{kk} = 1,\ a_{ki} = 0,\ \text{if } 1 \leqslant i \leqslant k-1\bigr\},\\
A'_k &= \bigl\{\bigl(a'_{ij}\bigr) \in G_k \mid a'_{11} = 1,\ a'_{i1} = 0,\ \text{if } 2 \leqslant i \leqslant k\bigr\},\\
C_k &= A_k \cap A'_k.
\end{aligned}$$

Let $A = A_n$, $A' = A'_n$ and $C = C_n$. Observe that

$$C = A \cap A' = \left\{ \begin{pmatrix} 1 & * \;\; \dots \;\; * & * \\ 0 & & * \\ \vdots & \boxed{\quad G_{n-2} \quad} & \vdots \\ 0 & & * \\ 0 & 0 \;\; \dots \;\; 0 & 1 \end{pmatrix} \right\}. \tag{2}$$

Throughout this paper, let $\theta : \mathbb{F}_q^+ \to \mathbb{C}^*$ be a fixed nontrivial homomorphism. Let $U = U_n$. Define a linear character $\Psi : U \to \mathbb{C}^*$ by

$$\Psi\bigl((u_{ij})\bigr) = \theta\left(\sum_{i=1}^{n-1} u_{i\,i+1}\right).$$

The linear character Ψ is in general position in the sense that the restriction of Ψ to each fundamental root subgroup X_{ii+1} is nontrivial. The induced character Ψ^G is called the *Gel'fand–Graev character* of G. It is well-known that all the irreducible components of a Gel'fand–Graev character occur with multiplicity 1 (see [2]).

For each $w \in W$, we define $l(w)$ to be the length of an expression of w as a product of generators $s \in S$. Let w_J be the longest element of $W_J = \langle J \rangle$ for each $J \subseteq S$. In particular, w_S is the longest element of W.

For each $w \in W$, let

$$\mathrm{Neg}(w) = \{\alpha \in \Phi^+ \mid w(\alpha) \in \Phi^-\},$$
$$\mathrm{Pos}(w) = \{\alpha \in \Phi^+ \mid w(\alpha) \in \Phi^+\}.$$

For each $w \in W$, we define the linear character $\Psi^w : U^w \to \mathbb{C}$ by $\Psi^w(u) = \Psi(wuw^{-1})$ for all $u \in U^w = w^{-1}Uw$. Let Ψ_w be the restriction of Ψ^w to the subgroup $U_w = U \cap U^w$ of U^w. So for all $\alpha \in U_w = \prod_{\alpha \in \mathrm{Pos}(w)} X_\alpha$ and $t \in \mathbb{F}_q$,

$$\Psi_w\big(x_\alpha(t)\big) = \begin{cases} 1 & \text{if } w(\alpha) \in \Phi^+ \setminus \Pi, \\ \theta(t) & \text{if } w(\alpha) \in \Pi. \end{cases} \tag{3}$$

For each $w \in W$, define

$$e_w := \frac{1}{|U_w|} \sum_{u \in U_w} \Psi_w\big(u^{-1}\big)u. \tag{4}$$

Then e_w is a primitive idempotent in the group algebra of U_w affording Ψ_w.

Definition 2.1. For each k with $0 \leqslant k \leqslant n-1$, we define the elements σ_k and σ'_k of the Weyl group W of G by

$$\sigma_k = \begin{cases} s_1 s_2 \dots s_k & \text{if } 1 \leqslant k \leqslant n-1, \\ 1 & \text{if } k = 0, \end{cases}$$
$$\sigma'_k = \begin{cases} s_{n-1} s_{n-2} \dots s_{n-k} & \text{if } 1 \leqslant k \leqslant n-1, \\ 1 & \text{if } k = 0. \end{cases}$$

Note that $\sigma_k \in \langle s_1, s_2, \dots, s_{n-2} \rangle \subset A$ and $\sigma'_k \in \langle s_2, s_3, \dots, s_{n-1} \rangle \subset A'$, for all $0 \leqslant k \leqslant n-2$.

For each $k, n \in \mathbb{Z}^+$ with $0 \leqslant k \leqslant n$ define

$$d_k(x) = (h_{ij}) \in H_n \quad \text{where } h_{ij} = \begin{cases} 1 & \text{if } i = j \neq k \\ x & \text{if } i = j = k \end{cases} \big(\text{for all } x \in \mathbb{F}_q^*\big). \tag{5}$$

Let $D_k = \{d_k(x) \in H_n \mid x \in \mathbb{F}_q^*\}$.

The following theorem gives a collection of some well-known facts about irreducible cuspidal modules of G. The proof will be omitted. Further details can be found in [6].

Theorem 2.2. *Let M be an irreducible cuspidal module of G. Then*

(i) *M has a unique (up to scalars) vector b such that*

$$u.b = \Psi(u)b$$

for all $u \in U$. Thus $\langle b\rangle$ is the unique $\mathbb{C}U$-submodule of $\mathrm{Res}_U^G M$ affording Ψ.

(ii) *The restriction to A of M is irreducible and is isomorphic to $\mathrm{Ind}_U^A\langle b\rangle$, the $\mathbb{C}A$-module induced from the $\mathbb{C}U$-module $\langle b\rangle$.*

(iii) $\dim_G M = (q-1)(q^2-1)\dots(q^{n-1}-1)$ *(the number of cosets of U in A).*

The main result of this paper is as follows:

Theorem 2.3. *Let $n \geqslant 2$ be any fixed integer. Let σ_k, σ_k' be as in Definition 2.1, b as in Theorem 2.2 and $C, e_w, d_k(x)$ as in Eqs. (2), (4) and (5).*

(i) *For $k, l \in \{0, 1, \dots, n-2\}$ and $x, y \in \mathbb{F}_q^*$,*

$$\mathrm{Hom}_{\mathbb{C}C}\big(\mathbb{C}Cd_1(x)e_{\sigma_k^{-1}}d_1(x)^{-1}, \mathbb{C}Cd_n(y)e_{(\sigma_l')^{-1}}d_n(y)^{-1}\big) = 0$$

unless $k = l$ and $xy = 1$.

(ii) *Let M be an irreducible cuspidal module of G. For each k with $0 \leqslant k \leqslant n-2$ and for each $x \in \mathbb{F}_q^*$,*

$$\mathrm{Res}_C^G M \cong \bigoplus_{\substack{0\leqslant k\leqslant n-2\\ x\in\mathbb{F}_q^*}} \mathbb{C}Cd_1(x)\sigma_k b \cong \bigoplus_{\substack{0\leqslant k\leqslant n-2\\ x\in\mathbb{F}_q^*}} \mathbb{C}Cd_n\big(x^{-1}\big)\sigma_k' b$$

and

$$\mathbb{C}Cd_1(x)\sigma_k b \cong_{\mathbb{C}C} \mathbb{C}Cd_n\big(x^{-1}\big)\sigma_k' b.$$

Furthermore $\mathbb{C}Cd_1(x)\sigma_k b$ is multiplicity-free, with irreducible components parametrized by the constituents of the Gel'fand–Graev representation of G_k.

Corollary 2.4. *With above notation, $\mathrm{Res}_C^G M$ is multiplicity-free.*

We give the proof of Theorem 2.3(i) in Section 4, using calculations with idempotents in the group algebra of G, given in the next section. The proof of Theorem 2.3(ii), completed in the last section proceeds by using Theorem 2.2(ii) introduced in Section 6, Mackey's formula for the restriction of an induced module to a subgroup and Clifford's Theorem for the decomposition of modules induced from normal subgroups. The remainder of this paper is devoted to proving the details of this.

3. Calculations with idempotents

For each n and k, define $U(k,n) = \{(u_{ij}) \in U_n \mid u_{ij} = \delta_{ij}, 1 \leqslant i, j \leqslant k\}$ and $U'(k,n) = \{(u'_{ij}) \in U_n \mid u'_{ij} = \delta_{ij}, k \leqslant i, j \leqslant n\}$. Thus

$$U(k,n) = \left\{ \begin{pmatrix} \boxed{I_k} & & * \\ & 1 & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \right\},$$

$$U'(k,n) = \left\{ \begin{pmatrix} 1 & & & * \\ & \ddots & & \\ & & 1 & \\ 0 & & & \boxed{I_{n-k+1}} \end{pmatrix} \right\}.$$

Note that $U(1,n) = U'(n,n) = U_n$ and $U(n,n) = U'(1,n) = \{I_n\}$.

For each $k, n \in \mathbb{Z}^+$ with $0 \leqslant k \leqslant n$ define

$$T_{k,n} = \big\{(t_{ij}) \in U_n \mid t_{ij} = \delta_{ij},\ j \neq k,\ 1 \leqslant i, j \leqslant n\big\},$$
$$T'_{k,n} = \big\{\big(t'_{ij}\big) \in U_n \mid t'_{ij} = \delta_{ij},\ i \neq k,\ 1 \leqslant i, j \leqslant n\big\}.$$

Thus $T_{k,n}$ is the product of the root subgroups X_{ik} for $i < k$, and $T'_{k,n}$ the product of the root subgroups X_{kj} for $j > k$. For brevity we write $U(k) = U(k,n)$, $U'(k) = U'(k,n)$, $T_k = T_{k,n}$ and $T'_k = T'(k,n)$.

Definition 3.1. For any positive integer k with $1 \leqslant k \leqslant n$, we define linear characters $\Psi_{k,n} : U(k,n) \to \mathbb{C}^*$ and $\Psi'_{k,n} : U'(k,n) \to \mathbb{C}^*$ by

$$\Psi_{k,n}\big((u_{ij})\big) = \theta\left(\sum_{i=k}^{n-1} u_{ii+1}\right), \qquad \Psi'_{k,n}\big(\big(u'_{ij}\big)\big) = \theta\left(\sum_{i=1}^{k-1} u'_{ii+1}\right)$$

for all $(u_{ij}) \in U(k,n)$ and $(u'_{ij}) \in U'(k,n)$.

We make the convention that $\Psi_{n,n} = \Psi'_{1,n} = 1$ (the trivial character of the trivial group $\{I_n\}$).

Definition 3.2. Let k and n be positive integers such that $1 \leqslant k \leqslant n$ and $x \in \mathbb{F}_q^*$. For each x, k and n, we define linear characters

$$ {}^{x}\varphi_{k,n}: T_{k,n} \to \mathbb{C}^*, \qquad {}^{x}\varphi'_{k,n}: T'_{k,n} \to \mathbb{C}^*, $$

by

$$ {}^{x}\varphi_{k,n}\big((t_{ij})\big) = \theta\big(x^{-1}t_{1k}\big), \qquad {}^{x}\varphi'_{k,n}\big((t'_{ij})\big) = \theta\big(xt'_{kn}\big), $$

for all $(t_{ij}) \in T_{kn}$ and $(t'_{ij}) \in T'_{kn}$.

In the case $x = 1$ we write ${}^{x}\varphi_{k,n} = \varphi_{k,n}$ and ${}^{x}\varphi'_{k,n} = \varphi'_{k,n}$. It can be verified that

$$ {}^{d_1(x)}\varphi_{k,n} = {}^{x}\varphi_{k,n}, \qquad {}^{d_n(x)}\varphi'_{k,n} = {}^{x}\varphi'_{k,n}. \tag{6} $$

Definition 3.3. Let m be a non-negative integer such that $m + k \leqslant n$. Define an embedding $\eta^m_{k,n}: G_k \to G_n$ by

$$ \eta^m_{k,n}(g) = \begin{pmatrix} I_m & & 0 \\ & g & \\ 0 & & I_{n-m-k} \end{pmatrix} $$

for all $g \in G_k$.

We shall sometimes use notations such as $G^m_{k,n}$ for $\eta^m_{k,n}(G_k)$, $C^m_{k,n}$ for $\eta^m_{k,n}(C_k)$, Ψ_k for $\Psi_{k,n}$, Ψ'_k for $\Psi'_{k,n}$, ${}^{x}\varphi_k$ for ${}^{x}\varphi_{k,n}$ and so on.

Let $k, m \in \mathbb{Z}$ be such that $0 \leqslant k \leqslant k + m \leqslant n$. We define the following idempotents of the subgroups $\eta^m_{k,n}(U_k) = U^m_k$, $U(k)$, $U'(k)$, T_k, T'_k of G:

$$ \begin{aligned} \big[U^m_k\big] &:= \frac{1}{|U^m_k|} \sum_{u \in U^m_k} \Psi\big(u^{-1}\big)(u), \\ \big[U(k)\big] &:= \frac{1}{|U(k)|} \sum_{u \in U(k)} \Psi_k\big(u^{-1}\big)(u), \\ \big[U'(k)\big] &:= \frac{1}{|U'(k)|} \sum_{u \in U'(k)} \Psi'_k\big(u^{-1}\big)(u), \\ [T_k] &:= \frac{1}{|T_k|} \sum_{t \in T_k} \varphi_k\big(t^{-1}\big)(t), \\ \big[T'_k\big] &:= \frac{1}{|T'_k|} \sum_{t \in T'_k} \varphi'_k\big(t^{-1}\big)(t). \end{aligned} \tag{7} $$

Observe that for all $x \in \mathbb{F}^*_q$,

$$d_1(x)[T_k]d_1(x)^{-1} = \frac{1}{|T_k|}\sum_{t\in T_k} {}^{x}\varphi_k\bigl(t^{-1}\bigr)(t),$$

$$d_n(x)\bigl[T_k'\bigr]d_n(x)^{-1} = \frac{1}{|T_k'|}\sum_{t\in T_k'} {}^{x}\varphi_k'\bigl(t^{-1}\bigr)(t).$$

For any $x \in \mathbb{F}_q^*$, $x \neq 1$, we write

$$[T_k]_x = d_1(x)[T_k]d_1(x)^{-1}, \qquad \bigl[T_k'\bigr]_x = d_n(x)\bigl[T_k'\bigr]d_n(x)^{-1}.$$

Let σ_k and σ_k' be as in Definition 2.1. It can be verified that

$$\operatorname{Neg}\bigl(\sigma_k^{-1}\bigr) = \{\alpha_{12}, \alpha_{13}, \ldots, \alpha_{1\,k+1}\}, \tag{8}$$

$$\operatorname{Neg}\bigl(\bigl(\sigma_k'\bigr)^{-1}\bigr) = \{\alpha_{n-1\,n}, \alpha_{n-2\,n}, \alpha_{n-3\,n}, \ldots, \alpha_{n-k\,n}\}. \tag{9}$$

Lemma 3.4. *For any $k \in \mathbb{Z}$ with $0 \leqslant k \leqslant n-2$ and $x \in \mathbb{F}_q^*$, we have*

$$d_1(x)e_{\sigma_k^{-1}}d_1(x)^{-1} = \bigl[U(k+2)\bigr][T_{k+2}]_x\bigl[U_k^1\bigr] = \bigl[U_k^1\bigr]\bigl[U(k+2)\bigr][T_{k+2}]_x. \tag{10}$$

Proof. By (4),

$$e_{\sigma_k^{-1}} = \frac{1}{|U_{\sigma_k^{-1}}|}\sum_{u\in U_{\sigma_k^{-1}}} \Psi_{\sigma_k^{-1}}\bigl(u^{-1}\bigr)u.$$

By (8),

$$U_{\sigma_k^{-1}} = \prod_{\alpha\in \operatorname{Pos}(\sigma_k^{-1})} X_\alpha$$

$$= \left\{\left(\begin{array}{cccc|ccc} 1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ & \multicolumn{3}{c|}{} & 0 & \ldots & 0 \\ & \multicolumn{3}{c|}{U_k} & \vdots & & \vdots \\ & \multicolumn{3}{c|}{} & 0 & \ldots & 0 \\ & & & & 1 & \ldots & 0 \\ & & & & & \ddots & \vdots \\ 0 & & & & & & 1 \end{array}\right)\left(\begin{array}{c|ccc} & * & \ldots & * \\ & * & \ldots & * \\ I_{k+1} & \vdots & & \vdots \\ & * & \ldots & * \\ & 1 & \ldots & * \\ & & \ddots & \vdots \\ 0 & & & 1 \end{array}\right)\right\}.$$

Thus

$$U_{\sigma_k^{-1}} = U(k+1) \rtimes U_k^1 \tag{11}$$

as it can be verified that U_k^1 normalizes $U(k+1)$, for all k with $1 \leqslant k \leqslant n-2$. Furthermore,

$$U_{\sigma_k^{-1}} = U \cap U^{\sigma_k^{-1}} = U(k+2) \rtimes T_{k+2,n} \rtimes U_k^1.$$

For any $(u_{ij}) \in U(k+2) \rtimes T_{k+2} \rtimes U_k^1$, we have

$$\Psi\left(\sigma_k^{-1} d_1(x)^{-1}(u_{ij}) d_1(x)\sigma_k\right) = \theta\left(\sum_{i=2}^{k} u_{i,i+1}\right)\theta\left(x^{-1}u_{1,k+2}\right)\theta\left(\sum_{i=k+2}^{n-1} u_{i,i+1}\right) \tag{12}$$

since $\sigma_k^{-1}(\alpha_{1,k+2}) = \alpha_{k+1,k+2}$, $\sigma_k^{-1}(\alpha_{k+1,1}) = \alpha_{k,k+1}$, $\sigma_k^{-1}(\alpha_{i,i+1}) = \alpha_{i-1,i}$ for $2 \leqslant i \leqslant k$ and $\sigma_k^{-1}(\alpha_{i,i+1}) = \alpha_{i,i+1}$ for $k+2 \leqslant i \leqslant n-1$. □

With a similar argument together with the fact that

$$U_{(\sigma_k')^{-1}} = U'(n-k) \rtimes U_k^{n-k-1}, \tag{13}$$

the following lemma can be proved.

Lemma 3.5. *For any $k \in \mathbb{Z}$ with $0 \leqslant k \leqslant n-2$, we have*

$$\begin{aligned} d_n(x)e_{(\sigma_k')^{-1}} d_n(x)^{-1} &= \left[U'(n-k-1)\right]\left[T'_{n-k-1}\right]_x\left[U_k^{n-k-1}\right] \\ &= \left[U_k^{n-k-1}\right]\left[U'(n-k-1)\right]\left[T'_{n-k-1}\right]_x. \end{aligned}$$

Let $\Omega = S \cap C = \{s_i \mid 2 \leqslant i \leqslant n-2\}$ and $W_\Omega = W \cap C = \langle \Omega \rangle$.

Lemma 3.6. *Suppose that $k, l \in \{0, 1, \ldots, n-2\}$. For any $w \in W_\Omega$, $h \in H$ and $u_1, u_2 \in U$, we have*

$$e_{\sigma_k^{-1}} u_1 h w u_2 e_{(\sigma_l')^{-1}} = \kappa e_{\sigma_k^{-1}} h w e_{(\sigma_l')^{-1}}$$

for some $\kappa \in \mathbb{C}^*$.

Proof. By (3) and (4), $e_{\sigma_k^{-1}} u_1 h = \kappa_1 e_{\sigma_k^{-1}} h u_1'$, for some $\kappa_1 \in \mathbb{C}^*$ and some $u_1' \in U_{w_S \sigma_k^{-1}} = \prod_{\alpha \in \mathrm{Neg}(\sigma_k^{-1})} X_\alpha$. If $w \in W_\Omega$ and $\alpha \in \mathrm{Neg}(\sigma_k^{-1})$ then by (8) we see that $w^{-1}(\alpha) \in \Phi^+$. Hence $w^{-1}u_1' w \in U$ and so

$$e_{\sigma_k^{-1}} u_1 h w u_2 e_{(\sigma_l')^{-1}} = \kappa_1 e_{\sigma_k^{-1}} h w u_3 e_{(\sigma_l')^{-1}} \tag{14}$$

for some $u_3 \in U$. By (3) and (4), we have $u_3 e_{(\sigma_l')^{-1}} = \kappa_2 u_3' e_{(\sigma_l')^{-1}}$ for some $\kappa_2 \in \mathbb{C}^*$ and $u_3' \in U_{w_S(\sigma_l')^{-1}} = \prod_{\alpha \in \mathrm{Neg}((\sigma_l')^{-1})} X_\alpha$. Now since $w \in W_\Omega$ and $\alpha \in \mathrm{Neg}((\sigma_l')^{-1})$, by (8),

$w(\alpha) = \alpha_{in}$ for some $i \geqslant 2$ and so by (8) and (9) it follows that $w(\alpha) \notin \mathrm{Neg}(\sigma_k^{-1})$, whence $w(\alpha) \in \mathrm{Pos}(\sigma_k^{-1})$. Thus

$$wu_3' w^{-1} \in \prod_{\gamma \in \mathrm{Pos}(\sigma_k^{-1})} X_\gamma = U_{\sigma_k^{-1}}.$$

So $e_{\sigma_k^{-1}} hwu_3 e_{(\sigma_l')^{-1}} = \kappa' e_{\sigma_k^{-1}} hwe_{(\sigma_l')^{-1}}$ for some $\kappa' \in \mathbb{C}^*$, by (3) and (4). Combining this with (14) gives the result. $\square$

For each $w \in W$, we define $S(w) = \{\alpha \in \Phi^+ \mid w(\alpha) \in \Pi\}$, the set of positive roots made simple by w, and $S'(w) = \mathrm{Pos}(w) \setminus S(w)$. Clearly, $\mathrm{Pos}(w) = S(w) \cup S'(w)$. For each root $\alpha \in \Phi$, we define $\overline{X}_\alpha$ and $\widetilde{X}_\alpha$ to be respectively the idempotents of the trivial character of X_α and the character $x_\alpha(t) \to \theta(t)$. It follows that for each $w \in W$, the idempotent e_w can be written as

$$\prod_{\substack{\alpha \in S'(w) \\ \beta \in S(w)}} \overline{X}_\alpha \widetilde{X}_\beta. \tag{15}$$

By the Chevalley Commutator Formula (see [4, (69.3) Proposition]),

$$[X_\gamma, X_\beta] \subseteq \prod_{\alpha \in S'(w)} X_\alpha$$

for each $\beta, \gamma \in \mathrm{Pos}(w)$. Consequently, the positions of $\widetilde{X}_\beta$ factors in the above expression can be varied arbitrarily. Hence

$$e_w \overline{X}_\alpha = \overline{X}_\alpha e_w = e_w, \tag{16}$$

$$e_w \widetilde{X}_\beta = \widetilde{X}_\beta e_w = e_w \tag{17}$$

for all $\alpha \in S'(w)$ and $\beta \in S(w)$.

The following lemma will be useful in proving Lemma 3.8 and can be proved by induction on length of w.

Lemma 3.7. *Let $w \in W$. For any sequence $\alpha_1, \alpha_2, \ldots, \alpha_k$ of distinct positive roots,*

$$U^w \cap (X_{\alpha_1} X_{\alpha_2} \ldots X_{\alpha_k}) = X_{\beta_1} X_{\beta_2} \ldots X_{\beta_m}$$

where $\beta_1, \beta_2, \ldots, \beta_m$ is the subsequence of $\alpha_1, \alpha_2, \ldots, \alpha_k$ consisting of those α_i such that $w(\alpha_i)$ is positive.

Lemma 3.8. *Let $w_1, w_2 \in W$. For any $w \in W$, $e_{w_1} w e_{w_2} \neq 0$ if and only if*

$$\alpha \in S(w_1) \cap w\big(S(w_2)\big)$$

whenever $\alpha \in \mathrm{Pos}(w_1) \cap \mathrm{Pos}(w^{-1}) \cap \mathrm{Pos}(w_2 w^{-1})$. *Equivalently,* $e_{w_1} w e_{w_2} \neq 0$ *if and only if*

$$\beta \in S(w_2) \cap w^{-1}\big(S(w_1)\big)$$

whenever $\beta \in \mathrm{Pos}(w_2) \cap \mathrm{Pos}(w) \cap \mathrm{Pos}(w_1 w)$.

Proof. For any $w \in W$, $e_{w_1} w e_{w_2} \neq 0$ if and only if $\Psi_{w_1}(u) = \Psi_{w_2}(w^{-1} u w)$ for all $u \in U_{w_1} \cap w U_{w_2} w^{-1}$. By Lemma 3.7, we have

$$U_{w_1} \cap w U_{w_2} w^{-1} = U \cap U^{w_1} \cap w\big(U \cap U^{w_2}\big) w^{-1} = U_{w_1} \cap U_{w^{-1}} \cap U_{w_2 w^{-1}} = \prod_{\alpha \in \mathrm{Pos}(w_1) \cap \mathrm{Pos}(w^{-1}) \cap \mathrm{Pos}(w_2 w^{-1})} X_\alpha .$$

Hence $e_{w_1} w e_{w_2} \neq 0$ if and only if

$$\Psi_{w_1}\big(x_\alpha(t)\big) = \Psi_{w_2}\big(w^{-1} x_\alpha(t) w\big) \quad \text{for all } t \in \mathbb{F}_q \tag{18}$$

for all $\alpha \in \mathrm{Pos}(w_1) \cap \mathrm{Pos}(w^{-1}) \cap \mathrm{Pos}(w_2 w^{-1})$. The proof can be completed by using (1) and (3). The second part can be proved in a similar fashion. □

Corollary 3.9. *Let* $w_1, w_2 \in W$. *Then for any* $w \in W$, $e_{w_1} w e_{w_2} = 0$ *if there exists* $\alpha \in S(w_1)$ *such that* $w^{-1}(\alpha) \in S'(w_2)$ *or if there exists* $\beta \in S(w_2)$ *such that* $w(\beta) \in S'(w_1)$.

Lemma 3.10. *Let* $w_1, w_2, w \in W$ *and* $h \in H$. *Then* $e_{w_1} h w e_{w_2} \neq 0$ *if and only if* $e_{w_1} w e_{w_2} \neq 0$ *and* h *centralizes the root subgroup* X_β *for all* $\beta \in w(S(w_2)) \cap S(w_1)$.

This lemma can be proved by arguments similar to those used in the proof of Lemma 3.8.

By (8) and (9), we observe that

$$S\big(\sigma_k^{-1}\big) = \{\alpha_{23}, \alpha_{34}, \ldots, \alpha_{k\,k+1}, \alpha_{1\,k+2}, \alpha_{k+2\,k+3}, \ldots, \alpha_{n-1\,n}\}, \tag{19}$$

$$S\big(\big(\sigma_k'\big)^{-1}\big) = \{\alpha_{12}, \ldots, \alpha_{n-k-2\,n-k-1}, \alpha_{n-k-1\,n}, \alpha_{n-k\,n-k+1}, \ldots, \alpha_{n-2\,n-1}\}. \tag{20}$$

For each $k \in \mathbb{Z}$ with $0 \leqslant k \leqslant n-2$, define a subset Γ^k of the symmetric group W_n and a subset Δ_k of W_Ω, by

$$\Gamma^k = \big\{w \in W_n \mid w(1) = 1,\ w(n) = n,\ w(i) = k + i,\ 2 \leqslant i \leqslant n-k-1\big\},$$
$$\Delta_k = \big\{w \in W_\Omega \mid e_{\sigma_k^{-1}} w e_{(\sigma_k')^{-1}} \neq 0\big\}.$$

Lemma 3.11. *For each $k \in \mathbb{Z}$ with $0 \leqslant k \leqslant n-2$, we have*

$$\Delta_k \subseteq \Gamma^k.$$

Furthermore, if $0 \leqslant l \leqslant n-2$ and $l \neq k$ then $e_{\sigma_k^{-1}} w e_{(\sigma_l')^{-1}} = 0$ for all $w \in W_\Omega$.

Proof. Suppose that $k, l \in \{0, 1, \dots, n-2\}$ and $w \in W_\Omega$ with $e_{\sigma_k^{-1}} w e_{(\sigma_l')^{-1}} \neq 0$. Let $w \in W_n$. Note that $w(1) = 1$ and $w(n) = n$, since $w \in W_\Omega$. We use induction on j to show that $w(j) = k + j$ for all $j \in \{2, 3, \dots, n-l-1\}$. If $l = n-2$, this is vacuously true. By (20), $\alpha_{1k+2} \in S(\sigma_k^{-1})$. Then

$$w^{-1}(\alpha_{1k+2}) = \alpha_{1w^{-1}(k+2)} \in \Phi^+,$$

and by (9), we see that $\alpha_{1w^{-1}(k+2)} \notin \mathrm{Neg}((\sigma_l')^{-1})$. So we must have $\alpha_{1w^{-1}(k+2)} \in \mathrm{Pos}((\sigma_l')^{-1})$. Lemma 3.8 gives that $\alpha_{1w^{-1}(k+2)} \in S((\sigma_l')^{-1})$. By (20), $w^{-1}(k+2) = n$ if $l = n-2$, and $w^{-1}(k+2) = 2$ if $l \leqslant n-3$. If $l = n-2$ then $k+2 = w(n) = n$, and so $k = n-2 = l$. If $l \leqslant n-3$ then $w(2) = k+2$.

Suppose now that $l \leqslant n-3$, $j \in \{3, 4, \dots, n-l-1\}$ and that $w(m) = k+m$ for all $m \in \{2, 3, \dots, j-1\}$. In particular, $k+j-1 = w(j-1) \leqslant n-1$ since $w(n) = n$. By (19), $\alpha_{k+j-1k+j} \in S(\sigma_k^{-1})$. Now

$$w^{-1}(\alpha_{k+j-1k+j}) = \alpha_{w^{-1}(k+j-1)w^{-1}(k+j)} = \alpha_{j-1w^{-1}(k+j)}$$

and $j-1 < w^{-1}(k+j)$ since $w^{-1}(k+j) \neq w^{-1}(1) = 1$ and

$$w^{-1}(k+j) \notin \left\{w^{-1}(k+2), w^{-1}(k+3), \dots, w^{-1}(k+j-2)\right\} = \{2, 3, \dots, j-2\}.$$

By (9), $\alpha_{j-1w(k+j)} \notin \mathrm{Neg}((\sigma_l')^{-1})$ as $j-1 < n-l-2$. So we must have $\alpha_{j-1w^{-1}(k+j)} \in \mathrm{Pos}((\sigma_l')^{-1})$. Lemma 3.8 gives that $\alpha_{j-1w^{-1}(k+j)} \in S((\sigma_l')^{-1})$. Since $j-1 \leqslant n-l-2$, it follows from (20) that $w^{-1}(k+j) = j$, completing the induction. Since we have shown in particular that $n+k-l-1 = w(n-l-1) \neq w(n) = n$, it follows that $n+k-l-1 \leqslant n-1$. So $k \leqslant l$.

It remains to prove that $l \leqslant k$. Since $l \neq n-2$, by (19), $\alpha_{n+k-l-1n+k-l} \in S(\sigma_k^{-1})$, and

$$w^{-1}(\alpha_{n+k-l-1n+k-l}) = \alpha_{n-l-1w^{-1}(n+k-l)} \in \Phi^+$$

since $\{1, 2, \dots, n-l-2\} = \{w^{-1}(1)\} \cup \{w^{-1}(k+2), \dots, w^{-1}(n+k-l-2)\}$. By (8), we see that $\alpha_{n-l-1w^{-1}(n+k-l)} \notin \mathrm{Neg}((\sigma_l')^{-1})$ and so by Lemma 3.8 $\alpha_{n-l-1w^{-1}(n+k-l)} \in S((\sigma_l')^{-1})$. By (20), $w(n+k-l) = n$. Hence $n+k-l = n$, and so $k = l$, as required. □

For any $k, m \in \mathbb{Z}$ with $1 \leqslant k \leqslant n-2$ and $0 \leqslant m \leqslant n-k$, define

$$J_k^m := \{s_{m+1}, s_{m+2}, \dots, s_{m+k-1}\}.$$

Let $J_k^0 = J_k$. Then $W_{J_k^m} = \langle J_k^m \rangle$,

$$\Pi_{J_k^m} = \{\alpha_{i\,i+1} \mid m+1 \leqslant i \leqslant m+k-1\},$$

$\Phi_{J_k^m} = W_{J_k^m}(\Pi_{J_k^m})$, $\Phi^+_{J_k^m} = \Phi^+ \cap \Phi_{J_k^m}$ and $\Phi^-_{J_k^m} = \Phi^- \cap \Phi_{J_k^m}$. Observe that

$$\left[U_k^m\right] = \prod_{\substack{\alpha \in \Phi_{J_k^m} \setminus \Pi_{J_k^m} \\ \beta \in \Pi_{J_k^m}}} \overline{X}_\alpha \widetilde{X}_\beta. \tag{21}$$

We need the following elementary result in proving Lemma 3.13 (see page 262 of [2]).

Lemma 3.12. *For each $J \subseteq S$ let w_J be the element of maximal length in W_J. If $w \in W$ satisfies $w(\Pi_J) \subseteq \Pi$ and $w(\Pi \setminus \Pi_J) \subseteq \Phi^-$ (for some $J \subseteq S$), then $w = w_S w_J$.*

Lemma 3.13. *Given k, m as above, define*

$$\Lambda_k^m := \left\{ w \in W_{J_k^m} \;\middle|\; \left[U_k^m\right] w \left[U_k^m\right] \neq 0 \right\}.$$

Then

$$\Lambda_k^m = \left\{ w_{J_k^m} w_J \mid J \subseteq J_k^m \right\}$$

where w_J is the longest element of W_J for any $J \subseteq J_k^m$.

Proof. Let $w \in \Lambda_k^m$ and define

$$\Pi(w) = \left\{ \alpha \in \Pi_{J_k^m} \mid w(\alpha) \in \Phi^+_{J_k^m} \right\}.$$

Then $\Pi(w) = \Pi_J$ for some $J \subseteq J_k^m$. If $\alpha \in \Pi_J$ then

$$\left[U_k^m\right] \widetilde{X}_{w(\alpha)} = \left[U_k^m\right] w \widetilde{X}_\alpha \neq 0$$

(since $w \in \Lambda_k^m$), and since $w(\alpha) \in \Phi^+_{J_k^m}$ it follows that $w(\alpha) \in \Pi_{J_k^m}$. Thus

$$\Pi_J = \left\{ \alpha \in \Pi_{J_k^m} \mid w(\alpha) \in \Pi_{J_k^m} \right\}.$$

It follows that

$$w(\Pi_J) = \left\{ \alpha \in \Pi_{J_k^m} \mid w^{-1}(\alpha) \in \Pi_{J_k^m} \right\} \subseteq \Pi_{J_k^m},$$

and so $w(\Pi_J) = \Pi_K$ for some $K \subseteq J_k^m$. By the definition of $\Pi(w)$, $w(\Pi_{J_k^m} \setminus \Pi_J) \subseteq \Phi^-$. Thus

$$w(\Pi_J) \subseteq \Pi_{J_k^m} \quad \text{and} \quad w(\Pi_{J_k^m} \setminus \Pi_J) \subseteq \Phi^-_{J_k^m}$$

and so $w = w_{J_k^m} w_J$, by Lemma 3.12.

Conversely, let $w \in W_{J_k^m}$ be such that $w = w_{J_k^m} w_J$ for some $J \subseteq J_k^m$. Since $w_J(\Pi_J) = \{-\alpha \mid \alpha \in \Pi_J\}$, it follows that

$$w_J(\Pi_J) = \bigl\{-w_{J_k^m}(\alpha) \mid \alpha \in \Pi_J\bigr\} \subseteq \bigl\{-w_{J_k^m}(\alpha) \mid \alpha \in \Pi_{J_k^m}\bigr\} = \Pi_{J_k^m},$$

and so $w(\Pi_J) = \Pi_K$ where $K \subseteq J_k^m$. Furthermore,

$$w\bigl(\Phi^+_{J_k^m} \setminus \Phi_J\bigr) = \Phi^-_{J_k^m} \setminus \Phi_K.$$

So $U_k^m \cap wU_k^m w^{-1} = \prod_{\alpha \in \Phi_K^+} X_\alpha$ and the characters Ψ and ${}^w\Psi$ agree on this subgroup. So $[U_k^m]w[U_k^m] \neq 0$. $\square$

Let $I(k) = J_k^1 = \{s_2, \ldots, s_k\}$ and $J(k) = J_k^{n-k-1} = \{s_{n-k}, \ldots, s_{n-2}\}$, for any $k \in \mathbb{Z}$ with $1 \leqslant k \leqslant n-2$. Note that $I(1)$ and $J(1)$ are empty. For each such k, define $w_k \in W_\Omega$ by

$$w_k(j) = \begin{cases} k+j & \text{if } 2 \leqslant j \leqslant n-k-1, \\ 2+j-n+k & \text{if } n-k \leqslant j \leqslant n-1. \end{cases} \tag{22}$$

Clearly, $w_k \in \Gamma^k$ and w_{n-2} is the identity permutation. Also we have

$$w_k^{-1}(\Pi_{I(k)}) = \Pi_{J(k)}. \tag{23}$$

Let $k \in \mathbb{Z}$ be such that $1 \leqslant k \leqslant n-2$. For each such k and each $J \subseteq J(k)$, define

$$w_{k,J} := w_k w_{J(k)} w_J.$$

Proposition 3.14. *Let $k \in \mathbb{Z}$ with $1 \leqslant k \leqslant n-2$. Then*

$$\Delta_k = \bigl\{w_{k,J} \mid J \subseteq J(k)\bigr\}.$$

Proof. By Lemma 3.11, each $w \in \Delta_k$ can be written as $w_k w'$ for some $w' \in W_{J(k)}$. Now we claim that $l(w) = l(w_k) + l(w')$. In fact, w_k is the unique element of minimal length in the coset $wW_{J(k)}$, because $w_k(\Pi_{J(k)}) \subseteq \Phi^+$ by (23). Then we have by (21), (1) and (23),

$$\bigl[U_k^1\bigr]w_k = w_k \prod_{\substack{\alpha \in \Phi_{J(k)} \setminus \Pi_{J(k)} \\ \beta \in \Pi_{J(k)}}} \overline{X}_\alpha \widetilde{X}_\beta = w_k\bigl[U_k^{n-k-1}\bigr].$$

Lemmas 3.4 and 3.5 give that

$$\begin{aligned} e_{\sigma_k^{-1}} w_k w' e_{(\sigma_k')^{-1}} &= \bigl[U(k+2)\bigr][T_{k+2}]\bigl[U_k^1\bigr]w_k w'\bigl[U'(n-k-1)\bigr]\bigl[T'_{n-k-1}\bigr]\bigl[U_k^{n-k-1}\bigr] \\ &= \bigl[U(k+2)\bigr][T_{k+2}]w_k\bigl(\bigl[U_k^{n-k-1}\bigr]w'\bigl[U_k^{n-k-1}\bigr]\bigr) \\ &\quad \times \bigl[U'(n-k-1)\bigr]\bigl[T'_{n-k-1}\bigr]. \end{aligned}$$

By Lemma 3.13, $w' = w_{J(k)} w_J$ for some $J \subseteq J(k)$.

Conversely, suppose that $w = w_k w'$, where $w' = w_{J(k)} w_J$ for some $J \subseteq J(k)$. Note that also $w = w'' w_k$ where $w'' = w_{I(k)} w_I$ for some $I \subseteq I(k)$. Now using (8), (9) and (19), we find that

$$\begin{aligned}&\operatorname{Pos}\bigl(\sigma_k^{-1}\bigr) \cap \operatorname{Pos}\bigl(w^{-1}\bigr) \cap \operatorname{Pos}\bigl(\bigl(\sigma_k'\bigr)^{-1} w^{-1}\bigr) \cap S\bigl(\sigma_k^{-1}\bigr)\\ &\quad = \{\alpha_{k+2\,k+3}, \ldots, \alpha_{n-1n}\} \cup I \cup \{\alpha_{1k+2}\}.\end{aligned}$$

Denote this set by I'. Similarly,

$$\begin{aligned}&w^{-1}\bigl(\operatorname{Pos}\bigl(\sigma_k^{-1}\bigr) \cap \operatorname{Pos}\bigl(w^{-1}\bigr) \cap \operatorname{Pos}\bigl(\bigl(\sigma_k'\bigr)^{-1} w^{-1}\bigr)\bigr) \cap S\bigl(\bigl(\sigma_k'\bigr)^{-1}\bigr)\\ &\quad = \{\alpha_{12}, \alpha_{23}, \ldots, \alpha_{n-k-2\,n-k-1}\} \cup J \cup \{\alpha_{n-k-1\,n}\}.\end{aligned}$$

Denote this set by J'. Since $w^{-1}(I') = J'$, it follows from Lemma 3.8 that $e_{\sigma_k^{-1}} w e_{(\sigma_k')^{-1}} \neq 0$, and so $w \in W_k$. $\square$

Recall that for all $n \in \mathbb{Z}$ and $k, m \in \{0, 1, \ldots, n\}$ with $m + k \leqslant n$, $H_k^m = \eta_{k,n}^m(H_k)$. Note that

$$H_k^m = \bigl\{d_{m+1}(y_1) \ldots d_{m+k}(y_k) \mid y_i \in \mathbb{F}_q^*, 1 \leqslant i \leqslant k\bigr\} = D_{m+1} D_{m+2} \ldots D_{m+k},$$

by (5). For $i, j, l \in \{1, 2, \ldots, n\}$, $j \neq n$, $y_i, y_l \in \mathbb{F}_q^*$ and $t \in \mathbb{F}_q$, $d_i(y_i^{-1}) x_{j\,j+1}(t) d_l(y_l) = x_{j\,j+1}(t)$ holds for

$$\begin{cases} \text{all } y_i, y_l & \text{if } i \neq j \text{ and } l \neq j+1,\\ y_i = y_{i+1} & \text{if } i = j \text{ and } l = j+1,\\ y_i = 1 & \text{if } i = j \text{ and } l \neq j+1,\\ y_{i+1} = 1 & \text{if } i \neq j \text{ and } l = j+1.\end{cases} \tag{24}$$

Lemma 3.15. *For each $k \in \mathbb{Z}$ with $1 \leqslant k \leqslant n-2$ and each $J \subseteq J(k)$, define*

$$H(k, J) := \bigl\{h \in D_2 D_3 \ldots D_n \mid e_{\sigma_k^{-1}} h w_{k,J} e_{(\sigma_k')^{-1}} \neq 0\bigr\}.$$

Then

$$\begin{aligned}H(k, J) = \bigl\{&d_2(y_2) d_3(y_3) \ldots d_{k+1}(y_{k+1}) \mid y_i = y_{i+1} \text{ if } \alpha_{i\,i+1} \in w_{k,J}(\Pi_J),\\ &2 \leqslant i \leqslant k\bigr\}.\end{aligned}$$

Proof. Lemma 3.10 gives that $h \in H(k, J)$ if and only if h centralizes X_α for all $\alpha \in S(\sigma_k^{-1}) \cap w_{k,J}(S((\sigma_k')^{-1}))$. By (20) and Lemma 3.11, we have

$$w_{k,J}\bigl(S\bigl(\bigl(\sigma_k'\bigr)^{-1}\bigr)\bigr) = \{\alpha_{1k+2}, \alpha_{k+2k+3}, \ldots, \alpha_{n-2n-1}, \alpha_{n-1n}\} \cup w_{k,J}(\Pi_{J(k)}).$$

Then $w_{J(k)}w_J(\Pi_J) \subseteq \Pi_{J(k)}$ and $w_{J(k)}w_J(\Pi_{J(k)} \setminus \Pi_J) \subseteq \Phi^-_{J(k)}$ implies that

$$w_{k,J}(\Pi_J) \subseteq w_k(\Pi_{J(k)}) = \Pi_{I(k)}, \qquad w_{k,J}(\Pi_{J(k)} \setminus \Pi_J) \subseteq \Phi^-,$$

by (23). Hence by (19), we obtain that

$$w_{k,J}\big(S\big((\sigma'_k)^{-1}\big)\big) \cap S\big(\sigma_k^{-1}\big) = \{\alpha_{1k+2}, \alpha_{k+2k+3}, \dots, \alpha_{n-2n-1}, \alpha_{n-1n}\} \cup w_{k,J}(\Pi_J).$$

Let $h = d_2(y_2)d_3(y_3)\dots d_n(y_n) \in H^1_{n-1}$ where $y_i \in \mathbb{F}_q^*$ for $2 \leqslant i \leqslant n$. For any $t \in \mathbb{F}_q$, $h^{-1}x_{1k+2}(t)h = x_{1k+2}(t)$ if and only if $y_{k+2} = 1$ by (24). For $k+2 \leqslant i \leqslant n-1$,

$$h^{-1}x_{ii+1}(t)h = x_{ii+1}(t)$$

if and only if $y_i = y_{i+1}$. For any $\alpha_{ii+1} \in w_k w_{J(k)} w_J(\Pi_J)$, $h^{-1}x_{ii+1}(t)h = x_{ii+1}(t)$ for all $t \in \mathbb{F}_q$ if and only if $y_i = y_{i+1}$. Thus $h \in H(k,J)$ if and only if $y_i = y_{i+1}$ whenever $\alpha_{ii+1} \in w_{k,J}(\Pi_J)$ and $y_i = 1$ for $k+2 \leqslant i \leqslant n$, as required. □

4. Proof of Theorem 2.3(i)

Note that $d_1(x)e_{\sigma_k^{-1}}d_1(x)^{-1}$ and $d_n(y)e_{(\sigma'_l)^{-1}}d_n(y)^{-1}$ are idempotents in the group algebra $\mathbb{C}C$. By Lemma 26.7 [5], the space of homomorphisms between the left ideals that they generate is isomorphic to

$$d_1(x)e_{\sigma_k^{-1}}d_1(x)^{-1}\mathbb{C}C d_n(y)e_{(\sigma'_l)^{-1}}d_n(y)^{-1}. \tag{25}$$

By (2) and the Bruhat decomposition of $\mathrm{GL}_{n-2}(q)$, each element of C can be written in the form u_1hwu_2 where $u_1, u_2 \in U$, $h \in D_2D_3\dots D_{n-1}$ and $w \in W_\Omega$. Then the space (25) is spanned by elements of the form

$$d_1(x)e_{\sigma_k^{-1}}d_1(x)^{-1}u_1hwu_2d_n(y)e_{(\sigma'_l)^{-1}}d_n(y)^{-1}$$

where $u_1, u_2 \in U$, $h \in D_2D_3\dots D_{n-1}$ and $w \in W_\Omega$. By Lemma 3.6, such an element is nonzero if and only if

$$e_{\sigma_k^{-1}}d_1(x)^{-1}hwd_n(y)e_{(\sigma'_l)^{-1}} \neq 0.$$

Note that here w commutes with $d_n(y)$ since $w \in W_\Omega$. Multiplying through by $d_1(x)d_2(x)\dots d_n(x)$ (which is central in G) and applying Lemma 3.11, Proposition 3.14 and Lemma 3.15 we see that the above element is nonzero only if $l = k$, $w \in W_k$ and

$$hd_2(x)d_3(x)\dots d_{n-1}(x)d_n(xy) \in H(k,J)$$

for some $J \subseteq J(k)$. Since this last condition implies that $xy = 1$, the result follows.

5. Cuspidal modules of finite general linear groups

The group $G = G_n$ has a left action on V, the vector space of n-component column vectors over $\mathbb{F}_q$. If E is any affine hyperplane of V not containing the zero vector, the subgroup

$$\mathcal{A} = \mathrm{Stab}_G(E) = \{g \in G \mid gE \subseteq E\} \tag{26}$$

is called the *affine subgroup* of G determined by E. In particular, if E is the affine hyperplane $x_n = 1$ then $\mathcal{A} = A$ $(= A_n)$. Similarly, G has a right action on the space of n-component of row vectors over $\mathbb{F}_q$. The setwise stabilizer of a hyperplane with respect to this action will also be called an affine subgroup. In particular, A' is the stabilizer of the hyperplane $\{(1, *, \ldots, *)\}$. The analogue of Theorem 2.2 is valid in this setting. Moreover, Part (ii) of Theorem 2.2 implies that $\mathrm{Ind}_U^A\langle b\rangle$ and $\mathrm{Ind}_U^{A'}\langle b\rangle$ are irreducible.

We write $Y = \langle b\rangle$. So Y affords Ψ. Following the terminology of [1], nonzero elements of Y are called *Bessel vectors*. Note that Y^G affords the Gel'fand–Graev character of G.

For the case $n = 1$, $G_1 = \mathbb{F}_q^*$ and every irreducible module is 1-dimensional. We are interested in irreducible cuspidal modules for $n \geqslant 2$.

The above discussion gives the following corollary of Theorem 2.2.

Corollary 5.1. *Let M be any irreducible cuspidal module. For a fixed $n \geqslant 2$,*

$$M|_C \cong Y^A|_C \cong Y^{A'}|_C.$$

Clearly, $M|_C$ is not irreducible (since $\mathrm{Res}_C^A \mathrm{Ind}_U^A\langle b\rangle$ has a submodule $\mathrm{Ind}_U^C\langle b\rangle$). We are interested in decomposing $M|_C$.

Applying Mackey's subgroup theorem to $Y^A|_C$ and $Y^{A'}|_C$ gives the following two decompositions of $M|_C$

$$Y^A|_C \cong \bigoplus_a \left({}^aY|_{{}^aU\cap C}\right)^C \quad \text{and} \quad Y^{A'}|_C \cong \bigoplus_{a'} \left({}^{a'}Y|_{{}^{a'}U\cap C}\right)^C, \tag{27}$$

where a and a' run through a set of double coset representatives of (C, U) in A and in A', respectively.

6. Double cosets

Lemma 6.1. *For any fixed integer $n \geqslant 2$, the following statements hold*:

(i) $G = \bigcup_{0\leqslant k\leqslant n-1} A' D_1 \sigma_k U$,
(ii) $G = \bigcup_{0\leqslant k\leqslant n-1} A D_n \sigma'_k U$.

Proof. It can be seen that

$$W = \bigcup_{0 \leqslant k \leqslant n-1} \sigma_k^{-1} \langle s_2, s_3, \ldots, s_{n-1} \rangle$$

since $\sigma_k^{-1} \langle s_2, s_3, \ldots, s_{n-1} \rangle$ consists of all permutation matrices where the first column has 1 in the $(k+1)$st row. Since A' is generated by U, $\langle s_2, s_3, \ldots, s_{n-1} \rangle$ and $D_2 D_3 \ldots D_n$, the Bruhat decomposition yields that

$$G = \bigcup_{0 \leqslant k \leqslant n-1} U \sigma_k^{-1} D_1 A'.$$

Taking inverses on both sides gives part (i). Part (ii) can be proved similarly.

Lemma 6.2. *For any fixed $n \geqslant 2$, the set*

$$D_A(C, U) = \left\{ d_1(x) \sigma_k \mid x \in \mathbb{F}_q^*,\ 0 \leqslant k \leqslant n-2 \right\}$$

is a set of representatives of the (C, U)-double cosets in A and the set

$$D_{A'}(C, U) = \left\{ d_n(x) \sigma_k' \mid x \in \mathbb{F}_q^*,\ 0 \leqslant k \leqslant n-2 \right\}$$

is a set of representatives of the (C, U)-double cosets in A'.

Proof. Part (i) of Lemma 6.1 for the case $n-1$ gives that

$$G_{n-1,n} := \eta^0_{n-1n}(G_{n-1}) = \bigcup_{\substack{0 \leqslant k \leqslant n-2 \\ x \in \mathbb{F}_q^*}} A'_{n-1,n} d_1(x) \sigma_k U_{n-1,n}$$

where n is any fixed positive integer with $n \geqslant 2$. Take the semidirect product by T_n on both sides. Observe that

$$C = \left\{ \begin{pmatrix} 1 & 0 & \ldots & 0 & * \\ 0 & & & & * \\ \vdots & & \boxed{\quad I_{n-2} \quad} & & \vdots \\ 0 & & & & * \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix} \right\} \rtimes \left\{ \begin{pmatrix} 1 & * & \ldots & * & 0 \\ 0 & & & & 0 \\ \vdots & & \boxed{\quad G_{n-2} \quad} & & \vdots \\ 0 & & & & 0 \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix} \right\} = T_n \rtimes A'_{n-1}, \tag{28}$$

$A = T_n \rtimes G_{n-1,n}$ and $U = T_n \rtimes U_{n-1,n}$. Hence

$$A = \bigcup_{\substack{0 \leqslant k \leqslant n-2 \\ x \in \mathbb{F}_q^*}} C d_1(x) \sigma_k U.$$

Similarly, using $C = T_1' \rtimes A^1_{n-1,n}$, $A' = T_1' \rtimes G^1_{n-1,n}$ and $U = T_1' \rtimes U^1_{n-1,n}$, applying part (ii) of Lemma 6.1 for the case $n-1$ and taking semidirect product by T_1' gives that

$$A' = \bigcup_{\substack{0\leqslant k\leqslant n-2\\ x\in\mathbb{F}_q^*}} C d_n(x)\sigma_k' U,$$

as required. □

Lemma 6.3. *Let $n \geqslant 2$ be fixed, $x \in \mathbb{F}_q^*$ and $k \in \mathbb{Z}^+$. If $1 \leqslant k \leqslant n-2$ then*

$$\begin{aligned} {}^{d_1(x)\sigma_k}U \cap C &= U(k+1) \rtimes U_k^1, \\ {}^{d_n(x)\sigma_k'}U \cap C &= U'(n-k) \rtimes U_k^{n-k-1}, \end{aligned}$$

and if $k=0$ then ${}^{d_1(x)}U \cap C = {}^{d_n(x)}\, U \cap C = U$.

Proof. First we observe that ${}^{d_1(x)\sigma_k}U = {}^{\sigma_k}U$ and ${}^{d_n(x)\sigma_k'}U = {}^{\sigma_k'}U$ for all $x \in \mathbb{F}_q^*$. Then the case for $k=0$ is obvious. Assume that k is an integer with $1 \leqslant k \leqslant n-2$. A straightforward argument using the decomposition of C into (U,U) double cosets establishes that ${}^{\sigma_k}U \cap C = {}^{\sigma_k}U \cap U$. The result follows from (11). □

7. Decomposition

By Lemmas 6.2 and 6.3, (27) becomes

$$M|_C \cong \bigoplus_{\substack{0\leqslant k\leqslant n-2\\ x\in\mathbb{F}_q^*}} \left({}^{d_1(x)\sigma_k}Y|_{U(k+1)\rtimes U_k^1}\right)^C \cong \bigoplus_{\substack{0\leqslant k\leqslant n-2\\ x\in\mathbb{F}_q^*}} \left({}^{d_n(x)\sigma_k'}Y|_{U'(n-k)\rtimes U_k^{n-k-1}}\right)^C. \quad (29)$$

Now the next task is to decompose each summand appeared in the above decompositions into irreducibles.

For each $n \geqslant 1$ and $k \in \mathbb{Z}$ with $1 \leqslant k \leqslant n$, define $Y_{k,n}$, $Y_{k,n}'$, $\mathcal{Y}_{k,n}$ and $\mathcal{Y}_{k,n}'$ to be, respectively, a $\mathbb{C}U(k,n)$-module affording $\Psi_{k,n}$, a $\mathbb{C}U'(k,n)$-module affording $\Psi_{k,n}'$, a $\mathbb{C}T_{k,n}$-module affording $\varphi_{k,n}$ and a $\mathbb{C}T_{k,n}'$-module affording $\varphi_{k,n}'$. We also write Y_k for $Y_{k,n}$ and similarly for the others.

For any fixed $n \geqslant 2$ and positive integers k, l, m such that $1 \leqslant k < l \leqslant n$ and $m + l \leqslant n$, the group $U(k,l)_l^m = \eta_{l,n}^m(U(k,l))$ is isomorphic to $U(k,l)$ and we denote by

$$\eta_{l,n}^m(Y_{k,l})$$

the $\mathbb{C}U(k,l)_l^m$-module obtained from the $\mathbb{C}U(k,l)$-module $Y_{k,l}$ by transport of structure. Similar notation will be used for the characters of these modules and the other modules defined above. Note that all the modules defined above are irreducible as they are 1-dimensional.

The following proposition is a consequence of Clifford's theorem on induction from normal subgroups and can be found in Chapter 11 of [3].

Proposition 7.1. *Let $\mathcal{G} = \mathcal{H} \rtimes \mathcal{K}$ be the semidirect product of an abelian normal subgroup $\mathcal{H}$ by a subgroup $\mathcal{K}$ of $\mathcal{G}$. Let $\mathcal{M}$ be an irreducible $\mathbb{C}\mathcal{H}$-module and $\mathcal{L} = I_{\mathcal{K}}(\mathcal{M})$ the inertia group of $\mathcal{M}$ in $\mathcal{K}$. Then the inertia group $I_{\mathcal{G}}(\mathcal{M}) = \mathcal{H}\mathcal{L}$ has an irreducible module $\widetilde{\mathcal{M}}$ which, as an $\mathbb{C}\mathcal{H}$-module, is isomorphic to $\mathcal{M}$ and as an $\mathbb{C}\mathcal{L}$-module is trivial. Furthermore,*

$$\operatorname{Ind}_{\mathcal{H}}^{\mathcal{G}}(\mathcal{M}) \cong \bigoplus_{\mathcal{N}} (\dim \mathcal{N}) \operatorname{Ind}_{I_{\mathcal{G}}(\mathcal{M})}^{\mathcal{G}}\big(\mathcal{N} \otimes \widetilde{\mathcal{M}}\big)$$

where $\mathcal{N}$ runs through a set of representatives of the isomorphism classes of irreducible $I_{\mathcal{G}}(\mathcal{M})/\mathcal{H}$-modules and each $\operatorname{Ind}_{I_{\mathcal{G}}(\mathcal{M})}^{\mathcal{G}}(\mathcal{N} \otimes \widetilde{\mathcal{M}})$ is irreducible.

Lemma 7.2. *Let $n \geqslant 2$ be any fixed integer. Then*

$$\operatorname{Ind}_{T_n}^{C}(Y_{n-1}) \cong \bigoplus_{\mathcal{N}} (\dim \mathcal{N}) \operatorname{Ind}_{T_n \rtimes C_{n-1}}^{C}\big(\eta_{n-1,n}^{0}(\mathcal{N}) \otimes \widetilde{Y}_{n-1}\big)$$

where $\mathcal{N}$ runs through a set of representatives of the isomorphism classes of irreducible $\mathbb{C}C_{n-1}$-modules. For each such $\mathcal{N}$,

$$\operatorname{Ind}_{T_n \rtimes C_{n-1}}^{C}\big(\eta_{n-1,n}^{0}(\mathcal{N}) \otimes \widetilde{Y}_{n-1}\big)$$

is irreducible. Similarly,

$$\operatorname{Ind}_{T_1'}^{C}\big(Y_2'\big) \cong \bigoplus_{\mathcal{N}} (\dim \mathcal{N}) \operatorname{Ind}_{T_1' \rtimes C_{n-1}^{1}}^{C}\big(\eta_{n-1,n}^{1}(\mathcal{N}) \otimes \widetilde{Y}_2'\big)$$

where $C_{n-1}^{1} = \eta_{n-1,n}^{1}(C_{n-1})$ and $\mathcal{N}$ runs through a set of representatives of the isomorphism classes of irreducible $\mathbb{C}C_{n-1}^{1}$-modules. For each such $\mathcal{N}$,

$$\operatorname{Ind}_{T_1' \rtimes C_{n-1}^{1}}^{C}\big(\eta_{n-1,n}^{1}\mathcal{N} \otimes \widetilde{Y}_2'\big)$$

is irreducible.

Proof. The case $n = 2$ is trivial; so assume that $n > 2$. By (28), C is the semidirect product of the abelian normal subgroup T_n by the group

$$A'_{n-1,n} = \left\{ \begin{pmatrix} 1 & * & \dots & * & 0 \\ 0 & & & & 0 \\ \vdots & & G_{n-2} & & \vdots \\ 0 & & & & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \right\},$$

and also that the action on Y_{n-1} of the element

$$t = \begin{pmatrix} 1 & 0 & \dots & 0 & t_{1n} \\ 0 & 1 & \dots & 0 & t_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & t_{n-1n} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \in T_n$$

is determined solely by t_{n-1n}. Thus the inertia group $I_{A'_{n-1,n}}(Y_{n-1})$ consists of the elements of $A'_{n-1,n}$ that preserve the sets $\{t \in T_n \mid t_{n-1n} = \lambda\}$ for all $\lambda \in \mathbb{F}_q$. Hence

$$I_{A'_{n-1,n}}(Y_{n-1}) = \left\{ \begin{pmatrix} 1 & * & \dots & * & * & 0 \\ 0 & & & & * & 0 \\ \vdots & & G_{n-3} & & \vdots & \vdots \\ 0 & & & & * & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix} \right\} = \eta^0_{n-1,n}(C_{n-1}).$$

Similarly, it can be seen that $I_{A^1_{n-1}}(Y'_2) = \eta^1_{n-1n}(C_{n-1}) = C^1_{n-1,n}$. Proposition 7.1 completes the proof. □

Lemma 7.3. *Let k, n be positive integers such that $1 \leqslant k \leqslant n$. Then*

$$\operatorname{Ind}^C_{U(k,n)}(Y_{k,n}) \cong \bigoplus_{\mathcal{N}} (\dim \mathcal{N}) \operatorname{Ind}^C_{U(k,n) \rtimes C_{k,n}} \big(\eta^0_{k,n}(\mathcal{N}) \otimes \widetilde{Y}_{k,n}\big)$$

where $\mathcal{N}$ runs through a set of representatives of the isomorphism classes of irreducible $\mathbb{C}C_k$-modules. For each such $\mathcal{N}$,

$$\operatorname{Ind}^C_{U(k,n) \rtimes C_{k,n}} \big(\eta^0_{k,n}(\mathcal{N}) \otimes \widetilde{Y}_{k,n}\big)$$

is irreducible. Similarly,

$$\operatorname{Ind}_{U'(n-k,n)}^{C}\big(Y'_{n-k,n}\big) \cong \bigoplus_{\mathcal{N}} (\dim\mathcal{N}) \operatorname{Ind}_{U'(n-k,n)\rtimes C_{k,n}^{n-k-1}}^{C}\big(\eta_{k,n}^{n-k-1}(\mathcal{N}) \otimes \widetilde{Y}'_{n-k,n}\big)$$

where $\mathcal{N}$ runs through a set of representatives of the isomorphism classes of irreducible $\mathbb{C}C_k$-modules. For each such $\mathcal{N}$,

$$\operatorname{Ind}_{U'(n-k,n)\rtimes C_{k,n}^{n-k-1}}^{C}\big(\eta_{k,n}^{n-k-1}(\mathcal{N}) \otimes \widetilde{Y}'_{n-k,n}\big)$$

is irreducible.

Proof. The proof is by induction on n. The case $n = 1$ is trivially true. The case $k = n \geqslant 1$ is also true as $U(n, n) = \{I_n\}$ and $\operatorname{Ind}_{U(k,n)}^{C}(Y_{k,n})$ is the regular module for C.

Proceeding inductively, suppose now that $n > 1$ and $k \leqslant n - 1$. By the inductive hypothesis,

$$\operatorname{Ind}_{U(k,n-1)}^{C_{n-1}}(Y_{k,n-1}) \cong \bigoplus_{\mathcal{N}} (\dim\mathcal{N}) \operatorname{Ind}_{U(k,n-1)\rtimes C_{k,n-1}}^{C_{n-1}}\big(\eta_{k,n-1}^{0}(\mathcal{N}) \otimes \widetilde{Y}_{k,n-1}\big) \tag{30}$$

where the summands on the right-hand side are irreducible. By Lemma 7.2, the map from $\mathbb{C}C_{n-1}$-modules to $\mathbb{C}C$-modules given by

$$\mathcal{M} \to \operatorname{Ind}_{T_n\rtimes C_{n-1,n}}^{C}\big(\eta_{n-1,n}^{0}(\mathcal{M}) \otimes \widetilde{Y}_{n-1}\big) \tag{31}$$

preserves irreducibility. Note that, in this formula, T_n acts trivially on $\eta_{n-1,n}^{0}(\mathcal{M})$; in other words, $\eta_{n-1,n}^{0}(\mathcal{M})$ is the $\mathbb{C}(T_n \rtimes C_{n-1,n})$-module obtained from $\mathcal{M}$ by lifting through the map $T_n \rtimes C_{n-1n} \to C_{n-1}$. Our desired conclusion will follow by applying the map (31) to both sides of (30). The preimage of $U(k, n-1) \subseteq C_{n-1}$ in $T_n \rtimes C_{n-1,n}$ is $U(k, n) = T_n \rtimes U(k, n-1)$; so

$$\eta_{n-1,n}^{0}\big(\operatorname{Ind}_{U(k,n-1)}^{C_{n-1}}(Y_{k,n-1})\big) \cong \operatorname{Ind}_{U(k,n)}^{T_n\rtimes C_{n-1,n}}\big(Y_{k,n-1}^{*}\big)$$

where $Y_{k,n-1}^{*}$ is $Y_{k,n-1}$ lifted to $U(k, n)$ via the map $U(k, n) \to U(k, n-1)$. Now

$$\begin{aligned}\operatorname{Ind}_{U(k,n)}^{T_n\rtimes C_{n-1,n}}\big(Y_{k,n-1}^{*}\big) \otimes \widetilde{Y}_{n-1} &\cong \operatorname{Ind}_{U(k,n)}^{T_n\rtimes C_{n-1,n}}\big(Y_{k,n-1}^{*} \otimes \big(\widetilde{Y}_{n-1}|_{U(k,n)}\big)\big)\\ &\cong \operatorname{Ind}_{U(k,n)}^{T_n\rtimes C_{n-1,n}}(Y_{k,n}).\end{aligned}$$

Thus applying the map (31) to the left-hand side of (30) gives

$$\operatorname{Ind}_{T_n\rtimes C_{n-1,n}}^{C}\big(\operatorname{Ind}_{U(k,n)}^{T_n\rtimes C_{n-1,n}}(Y_{k,n})\big) \cong \operatorname{Ind}_{U(k,n)}^{C}(Y_{k,n}).$$

Turning now to the right-hand side of (30), we observe that the preimage of $U(k, n-1) \rtimes C_{k,n-1}$ in $T_n \rtimes C_{n-1n}$ is $U(k, n) \rtimes C_{k,n}$ and so

$$\eta_{n-1,n}^{0}\big(\operatorname{Ind}_{U(k,n-1)\rtimes C_{k,n-1}}^{C_{n-1}}\big(\eta_{k,n-1}^{0}(\mathcal{N}) \otimes \widetilde{Y}_{k,n-1}\big)\big) \cong \operatorname{Ind}_{U(k,n)\rtimes C_{k,n}}^{T_n\rtimes C_{n-1,n}}\big(\eta_{k,n-1}^{0}(\mathcal{N}) \otimes \widetilde{Y}_{k,n-1}^{*}\big)$$

where $\widetilde{Y}^*_{k,n-1}$ is $\widetilde{Y}_{k,n-1}$ lifted to $U(k,n) \rtimes C_{k,n}$ via the map $U(k,n) \rtimes C_{k,n} \to U(k,n-1) \rtimes C_{k,n-1}$. Now since

$$\begin{aligned}
&\operatorname{Ind}^{T_n \rtimes C_{n-1,n}}_{U(k,n)\rtimes C_{k,n}}\big(\eta^0_{k,n}(\mathcal{N}) \otimes \widetilde{Y}^*_{k,n-1}\big) \otimes \widetilde{Y}_{n-1} \\
&\quad\cong \operatorname{Ind}^{T_n \rtimes C_{n-1,n}}_{U(k,n)\rtimes C_{k,n}}\big(\eta^0_{k,n}(\mathcal{N}) \otimes \widetilde{Y}^*_{k,n-1} \otimes \big(\widetilde{Y}_{n-1}|_{U(k,n)\rtimes C_{k,n}}\big)\big) \\
&\quad\cong \operatorname{Ind}^{T_n \rtimes C_{n-1,n}}_{U(k,n)\rtimes C_{k,n}}\big(\eta^0_{k,n}(\mathcal{N}) \otimes \widetilde{Y}_{k,n}\big),
\end{aligned}$$

it follows that applying the map (31) to the right-hand side of (30) gives

$$\bigoplus_{\mathcal{N}} (\dim \mathcal{N}) \operatorname{Ind}^{C}_{U(k,n)\rtimes C_{k,n}}\big(\eta^0_{k,n}(\mathcal{N}) \otimes \widetilde{Y}_{k,n}\big)$$

with all summands irreducible, as desired.

The second part can be proved by using the second part of Lemma 7.2 and adopting the same method as above. □

Lemma 7.4. *Let $n \geqslant 2$ be any fixed integer and $x \in \mathbb{F}_q^*$. Then*

$$\operatorname{Ind}^C_{T_n}\big({}^x\mathcal{Y}_n\big) \cong \bigoplus_{\mathcal{N}} (\dim \mathcal{N}) \operatorname{Ind}^C_{T_n \rtimes G^1_{n-2}}\big(\eta^1_{n-2,n}(\mathcal{N}) \otimes {}^x\widetilde{\mathcal{Y}}_n\big)$$

where $\mathcal{N}$ runs through a set of representatives of the isomorphism classes of irreducible $\mathbb{C}G_{n-2}$-modules. For each such $\mathcal{N}$,

$$\operatorname{Ind}^C_{T_n \rtimes G^1_{n-2}}\big(\eta^1_{n-2,n}(\mathcal{N}) \otimes {}^x\widetilde{\mathcal{Y}}_n\big)$$

is irreducible. Similarly,

$$\operatorname{Ind}^C_{T_1'}\big({}^x\mathcal{Y}_1'\big) \cong \bigoplus_{\mathcal{N}} (\dim \mathcal{N}) \operatorname{Ind}^C_{T_1' \rtimes G^1_{n-2}}\big(\eta^1_{n-2,n}(\mathcal{N}) \otimes {}^x\widetilde{\mathcal{Y}}_1'\big)$$

where $\mathcal{N}$ runs through a set of representatives of the isomorphism classes of irreducible $\mathbb{C}G_{n-2}$-modules. For each such $\mathcal{N}$,

$$\operatorname{Ind}^C_{T_n \rtimes G^1_{n-2}}\big(\eta^1_{n-2,n}(\mathcal{N}) \otimes {}^x\widetilde{\mathcal{Y}}_1'\big)$$

is irreducible.

Proof. By (28), the inertia group $I_C({}^x\mathcal{Y}_n) = T_n \rtimes G^1_{n-2}$. Also $I_C({}^x\mathcal{Y}_1') = T_1' \rtimes G^1_{n-2}$. This implies that

$$\begin{aligned}
I_{A'_{n-1}}\big({}^x\mathcal{Y}_n\big) &= I_C\big({}^x\mathcal{Y}_n\big) \cap A'_{n-1} = G^1_{n-2}, \\
I_{A^1_{n-1}}\big({}^x\mathcal{Y}_1'\big) &= I_C\big({}^x\mathcal{Y}_1'\big) \cap A^1_{n-1} = G^1_{n-1}.
\end{aligned}$$

The completion of the proof follows immediately from Proposition 7.1. □

Let $k, n \in \mathbb{Z}$ with $1 \leqslant k \leqslant n-2$ and let $x \in \mathbb{F}_q^*$. By (12)

$$^{d_1(x)\sigma_k}\Psi|_{U(k+2)\rtimes T_{k+2,n}\rtimes U_k^1} = {}^{\eta_k^1}\Psi_{1,k}{}^{x}\varphi_{k+2,n}\Psi_{k+2,n}.$$

The three factors on the right-hand side of this equation can all be regarded as characters of $U(k+2) \rtimes T_{k+2,n} \rtimes U_k^1$. For $\eta_k^1(\Psi_{1,k})$ this is accomplished via the homomorphism $U(k+2) \rtimes T_{k+2,n} \rtimes U_k^1 \to U_k^1$. The character $\Psi_{k+2,n}$ of $U(k+2)$ is invariant under the action of $T_{k+2,n} \rtimes U_k^1$, and hence extends by Proposition 7.1. In the notation of Lemma 7.3, this character is afforded by the restriction of the $\mathbb{C}(U(k+2) \rtimes C_{k+2})$-module $\widetilde{Y}_{k+2,n}$. Similarly, the homomorphism $U(k+1) = U(k+2) \rtimes T_{k+2,n} \to T_{k+2,n}$ lifts ${}^{x}\varphi_{k+2,n}$ to a character of $U(k+1)$ which is invariant under the action of G_k^1. The corresponding $\mathbb{C}(U(k+1) \rtimes G_k^1)$-module will be denoted by ${}^{x}\widetilde{\mathcal{Y}}_{k+2,n}$ (in agreement with the notation adopted in Lemma 7.4 for the case $k = n-2$).

The above discussion has proved the following result.

Lemma 7.5. *For a fixed $n \geqslant 2$, for any $x \in \mathbb{F}_q^*$ and any $k \in \mathbb{Z}$ with $0 \leqslant k \leqslant n-2$,*

$$^{d_1(x)\sigma_k}Y|_{U(k+1)\rtimes U_k^1} \cong \eta_k^1(Y_{1,k}) \otimes \left({}^{x}\widetilde{\mathcal{Y}}_{k+2,n}|_{U(k+1)\rtimes U_k^1}\right) \otimes \left(\widetilde{Y}_{k+2,n}|_{U(k+1)\rtimes U_k^1}\right)$$

where the module ${}^{x}\widetilde{\mathcal{Y}}_{k+2}$ is isomorphic to ${}^{x}\mathcal{Y}_{k+2}$ as $\mathbb{C}T_{k+2}$-module and is trivial as $\mathbb{C}(U(k+2) \rtimes U_k^1)$-module; similarly, the module $\widetilde{Y}_{k+2}$ is isomorphic to Y_{k+2} as $\mathbb{C}U(k+2)$-module and is trivial as $\mathbb{C}(T_{k+2} \rtimes U_k^1)$-module.

With similar arguments the following result can be proved.

Lemma 7.6. *Let $n \geqslant 2$ be any fixed integer. For any $x \in \mathbb{F}_q^*$ and any $k \in \{0, 1, \dots n-2\}$,*

$$\begin{aligned} ^{d_n(x)\sigma'_k}Y|_{U'(n-k)\rtimes U_k^{n-k-1}} \cong \eta_k^{n-k-1}(Y_{1,k}) &\otimes \left({}^{x}\widetilde{\mathcal{Y}}'_{n-k-1,n}|_{U'(n-k)\rtimes U_k^{n-k-1}}\right) \\ &\otimes \left(\widetilde{Y}'_{n-k-2,n}|_{U'(n-k)\rtimes U_k^{n-k-1}}\right) \end{aligned}$$

where ${}^{x}\widetilde{\mathcal{Y}}'_{n-k-1}$ is isomorphic to ${}^{x}\mathcal{Y}'_{n-k-1}$ as $\mathbb{C}T'_{n-k-1}$-module and is trivial as $\mathbb{C}(U'(n-k-2) \rtimes U_k^{n-k-1})$-module; similarly, $\widetilde{Y}'_{n-k-2}$ is isomorphic to Y'_{n-k-2} as $\mathbb{C}U'(n-k-2)$-module and is trivial as $\mathbb{C}\,(T'_{n-k-1} \rtimes U_k^{n-k-1})$-module.

Note that for any $x_1, x_2, \dots, x_n \in \mathbb{F}_q^*$

$$\left({}^{d_1(x_1)d_2(x_2)\dots d_n(x_n)}Y\right)^G \cong_{\mathbb{C}G} Y^G$$

as $d_i(x_i)$ normalizes U for all $1 \leqslant i \leqslant n$.

The following proposition is the key result in proving Theorem 2.3(ii).

Proposition 7.7. *Let $n \geqslant 2$ be fixed, $k \in \mathbb{Z}$ be such that $0 \leqslant k \leqslant n-2$. For any $x \in \mathbb{F}_q^*$, we have*

$$\operatorname{Ind}_{U(k+1)\rtimes U_k^1}^{C}\Bigl(\operatorname{Res}_{U(k+1)\rtimes U_k^1}^{d_1(x)\sigma_k U}\bigl({}^{d_1(x)\sigma_k}Y\bigr)\Bigr) \cong \bigoplus_{\mathcal{N}} \operatorname{Ind}_{U(k+1)\rtimes G_k^1}^{C}\bigl(\eta_{k,n}^1(\mathcal{N}) \otimes {}^x\widetilde{\mathcal{Y}}_{k+2,n} \otimes \widetilde{Y}_{k+2,n}\bigr)$$

where $\mathcal{N}$ runs through a set of representatives of the isomorphism classes of irreducible components of the Gel'fand–Graev module $Y_{1,k}^{G_k}$ for G_k. Furthermore, the summands on the right hand side are all irreducible. Similarly,

$$\operatorname{Ind}_{U'(n-k)\rtimes U_k^{n-k-1}}^{C}\Bigl(\operatorname{Res}_{U'(n-k)\rtimes U_k^{n-k-1}}^{d_n(x)\sigma_k' U}\bigl({}^{d_n(x)\sigma_k'}Y\bigr)\Bigr) \cong \bigoplus_{\mathcal{N}} \operatorname{Ind}_{U'(n-k)\rtimes G_k^{n-k-1}}^{C}\bigl(\eta_{k,n}^{n-k-1}(\mathcal{N}) \otimes {}^x\widetilde{\mathcal{Y}}'_{n-k-1,n} \otimes \widetilde{Y}'_{n-k-2,n}\bigr)$$

where $\mathcal{N}$ runs through a set of representatives of the isomorphism classes of irreducible components of the Gel'fand–Graev module $Y_{1,k}^{G_k}$ for G_k, the summands on the right-hand side being irreducible.

Proof. Assume that $k, n \in \mathbb{Z}$ with $0 \leqslant k \leqslant n-2$. For brevity we shall employ the exponential notation for induction, writing $Y_{1,k}^{G_k}$ for $\operatorname{Ind}_{U_k}^{G_k}(Y_{1,k})$, and so on. Let

$$Y_{1,k}^{G_k} \cong \bigoplus_{\mathcal{N}} \mathcal{N} \tag{32}$$

be the decomposition of $Y_{1,k}^{G_k}$ into irreducibles. Each irreducible component of $Y_{1,k}^{G_k}$ occurs with multiplicity 1 as the Gel'fand–Graev character is multiplicity free (see [2]). Let $x \in \mathbb{F}_q^*$. Applying the embedding $\eta_{k,k+2}^1 : G_k \to G_{k+2}$, tensoring both sides of (32) by ${}^x\widetilde{\mathcal{Y}}_{k+2,k+2}$ and then inducing to C_{k+2} gives

$$\bigl(\eta_{k,k+2}^1\bigl(Y_{1,k}^{G_k}\bigr) \otimes {}^x\widetilde{\mathcal{Y}}_{k+2,k+2}\bigr)^{C_{k+2}} \cong \bigoplus_{\mathcal{N}} \bigl(\eta_{k,k+2}^1(\mathcal{N}) \otimes {}^x\widetilde{\mathcal{Y}}_{k+2,k+2}\bigr)^{C_{k+2}}. \tag{33}$$

Applying Lemma 7.4 for $n = k+2$ implies that each term on the right-hand side of (33) is irreducible. By the transitivity of induction, the left-hand side of (33) is equal to

$$\bigl(\eta_{k,k+2}^1(Y_{1,k}) \otimes \bigl({}^x\widetilde{\mathcal{Y}}_{k+2,k+2}|_{T_{k+2,k+2}\rtimes U_{k,k+2}^1}\bigr)\bigr)^{C_{k+2}}.$$

By applying the embedding $\eta_{k+2,n}^0 : G_{k+2} \to G_n$ to (33), we obtain

$$\bigl(\eta_{k,n}^1(Y_{1,k}) \otimes {}^x\widetilde{\mathcal{Y}}_{k+2,n}|_{T_{k+2,n}\rtimes U_{k,n}^1}\bigr)^{C_{k+2,n}} \cong \bigoplus_{\mathcal{N}} \bigl(\eta_{k,n}^1(\mathcal{N}) \otimes {}^x\widetilde{\mathcal{Y}}_{k+2,n}\bigr)^{C_{k+2,n}}. \tag{34}$$

Tensoring (34) on both sides with $\widetilde{Y}_{k+2,n}$ and then inducing to C gives

$$\Big(\big(\eta^1_{k,n}(Y_{1,k}) \otimes {}^x\widetilde{\mathcal{Y}}_{k+2,n}|_{T_{k+2,n} \rtimes U^1_{k,n}}\big)^{C_{k+2,n}} \otimes \widetilde{Y}_{k+2,n}\Big)^C$$
$$\cong \left(\bigoplus_{\mathcal{N}} \big(\eta^1_{k,n}(\mathcal{N}) \otimes {}^x\widetilde{\mathcal{Y}}_{k+2,n}\big)^{C_{k+2,n}} \otimes \widetilde{Y}_{k+2,n}\right)^C.$$

It follows that

$$\big(\eta^1_{k,n}(Y_{1,k}) \otimes {}^x\widetilde{\mathcal{Y}}_{k+2,n}|_{U(k+2)\rtimes T_{k+2,n}\rtimes U^1_{k,n}} \otimes \widetilde{Y}_{k+2,n}|_{U(k+2)\rtimes T_{k+2,n}\rtimes U^1_{k,n}}\big)^C$$
$$\cong \bigoplus_{\mathcal{N}} \big(\eta^1_{k,n}(\mathcal{N}) \otimes {}^x\widetilde{\mathcal{Y}}_{k+2,n} \otimes \widetilde{Y}_{k+2,n}\big)^C. \tag{35}$$

Lemma 7.3 implies that, for each such $\mathcal{N}$,

$$\big(\big(\eta^1_{k,n}(\mathcal{N}) \otimes {}^x\widetilde{\mathcal{Y}}_{k+2,n}\big) \otimes \widetilde{Y}_{k+2,n}\big)^C$$

is irreducible. By Lemma 7.5, the left hand side of (35) is equal to

$$\big({}^{d_1(x)\sigma_k}Y|_{U(k+1)\rtimes U^1_k}\big)^C.$$

This completes the proof of the first part of the proposition.

In a similar way, the second part can be proven by using Lemma 7.6 instead of Lemma 7.5. □

Lemma 7.8. *Let $k \in \mathbb{Z}$ with $0 \leqslant k \leqslant n-2$. Let $\mathcal{N}$ be an irreducible component of the Gel'fand–Graev module $Y^{G_k}_{1,k}$. Then*

$$\big(\eta^1_{k,n}(\mathcal{N}) \otimes {}^x\widetilde{\mathcal{Y}}_{k+2,n} \otimes \widetilde{Y}_{k+2,n}\big)^C \cong \big(\eta^{n-k-1}_{k,n}(\mathcal{N}) \otimes {}^{x^{-1}}\widetilde{\mathcal{Y}}'_{n-k-1,n} \otimes \widetilde{Y}'_{n-k-2,n}\big)^C.$$

Proof. Let ρ be the character of G_k afforded by $\mathcal{N}$. Then $\eta^1_{k,n}(\mathcal{N}) \otimes {}^x\widetilde{\mathcal{Y}}_{k+2,n} \otimes \widetilde{Y}_{k+2,n}$ affords the character

$$\lambda_{1,k,x} := \eta^1_k(\rho)^x\varphi_{k+2,n}\Psi_{k+2,n}$$

and $\eta^{n-k-1}_{k,n}(\mathcal{N}) \otimes {}^{x^{-1}}\widetilde{\mathcal{Y}}'_{n-k-1,n} \otimes \widetilde{Y}'_{n-k-2,n}$ affords the character

$$\lambda_{2,k,x^{-1}} := \eta^{n-k-1}_k(\rho)^{x^{-1}}\varphi_{n-k-1,n}\Psi_{n-k-1,n}.$$

By Proposition 7.7, $\lambda^C_{1,k,x}$ and $\lambda^C_{2,k,x^{-1}}$ are irreducible. Hence $\lambda^C_{1,k,x} = \lambda^C_{2,k,x^{-1}}$ if $(\lambda^C_{1,k,x}, \lambda^C_{2,k,x^{-1}}) > 0$. By Intertwining Number Theorem [3, Section 10.24], we have that $(\lambda^C_{1,k,x}, \lambda^C_{2,k,x^{-1}}) > 0$ if and only if there exists $c \in C$ such that

$$\bigl(\lambda_{1,k,x}, {}^c\lambda_{2,k,x^{-1}}\bigr)\big|_{U(k+1)\rtimes G_k^1 \cap {}^c(U'(n-k)\rtimes G_k^{n-k-1})} \neq 0.$$

Now define

$$I(k,x) := \begin{pmatrix} 1 & & & 0 \\ & & I_k & \\ & xI_{n-k-2} & & \\ 0 & & & 1 \end{pmatrix}$$

for each $x \in \mathbb{F}_q^*$. For simplicity, let

$$\mathcal{L}_k = U(k+1) \rtimes G_k^1 \cap {}^{I(k,x^{-1})}\bigl(U'(n-k) \rtimes G_k^{n-k-1}\bigr).$$

By a straight forward calculation, it can be seen that

$$\mathcal{L}_k = \left\{ \begin{pmatrix} 1 & 0 & \dots & 0 & l_{1,k+2} & * & \dots & \dots & * & * \\ & \multicolumn{3}{c}{\boxed{\quad g \quad}} & 0 & 0 & \dots & \dots & 0 & 0 \\ & & & & \vdots & \vdots & & & \vdots & \vdots \\ & & & & 0 & 0 & \dots & \dots & 0 & 0 \\ & & & & 1 & l_{k+2,k+3} & \dots & & * & * \\ & & & & & & \ddots & & \vdots & \vdots \\ & & & & & & & & l_{n-2,n-1} & * \\ & & & & & & & & 1 & l_{n-1,n} \\ 0 & & & & & & & & & 1 \end{pmatrix} \;\middle|\; g \in G_k \right\}$$

and

$$\lambda_{1,k,x}|_{\mathcal{L}_k} = {}^{I(k,x^{-1})}\lambda_{2,k,x^{-1}}|_{\mathcal{L}_k}. \qquad \square$$

8. Proof of Theorem 2.3(ii)

Let M be any irreducible cuspidal module of G. By Proposition 7.7, (29) becomes

$$M|_C \cong \bigoplus_{\substack{0\leqslant k\leqslant n-2\\ x\in\mathbb{F}_q^*}} \left(\bigoplus_{\mathcal{N}} \bigl(\eta^1_{k,n}(\mathcal{N}) \otimes {}^x\widetilde{\mathcal{Y}}_{k+2,n} \otimes \widetilde{Y}_{k+2,n}\bigr)^C \right)$$

$$\cong \bigoplus_{\substack{0\leqslant k\leqslant n-2\\ x\in\mathbb{F}_q^*}} \left(\bigoplus_{\mathcal{N}} \bigl(\eta^{n-k-1}_{k,n}(\mathcal{N}) \otimes {}^x\widetilde{\mathcal{Y}}'_{n-k-1,n} \otimes \widetilde{Y}'_{n-k-2,n}\bigr)^C \right)$$

where $\mathcal{N}$ runs through a set of representatives of isomorphism classes of irreducible component of the Gel'fand–Graev module $Y_{1,k}^{G_k}$ of G_k and the summands are all irreducible. By Lemma 7.8,

$$\bigoplus_{\mathcal{N}}\left(\eta_{k,n}^{1}(\mathcal{N}) \otimes {}^{x}\widetilde{\mathcal{Y}}_{k+2,n} \otimes \widetilde{Y}_{k+2,n}\right)^{C} \cong_{\mathbb{C}C} \bigoplus_{\mathcal{N}}\left(\eta_{k,n}^{n-k-1}(\mathcal{N}) \otimes {}^{x^{-1}}\widetilde{\mathcal{Y}}'_{n-k-1,n} \otimes \widetilde{Y}'_{n-k-2,n}\right)^{C},$$

for each $k \in \mathbb{Z}$ with $0 \leqslant k \leqslant n-2$ and each $x \in \mathbb{F}_q^*$.

Therefore, it follows that for each $k \in \mathbb{Z}$, $0 \leqslant k \leqslant n-2$,

$$\left({}^{d_1(x)\sigma_k}Y|_{U(k+1)\rtimes U_k^1}\right)^C \cong \left({}^{d_n(x^{-1})\sigma'_k}Y|_{U'(n-k)\rtimes U_k^{n-k-1}}\right)^C.$$

As $Y = \langle b\rangle = \mathbb{C}b$, we have

$$\left({}^{d_1(x)\sigma_k}Y|_{U(k+1)\rtimes U_k^1}\right)^C \cong_{\mathbb{C}C} \mathbb{C}Cd_1(x)\sigma_k b,$$
$$\left({}^{d_n(x^{-1})\sigma'_k}Y|_{U'(n-k)\rtimes U_k^{n-k-1}}\right)^C \cong_{\mathbb{C}C} \mathbb{C}Cd_n\left(x^{-1}\right)\sigma'_k b$$

for all $x \in \mathbb{F}_q^*$ and $k \in \mathbb{Z}$ with $0 \leqslant k \leqslant n-2$. Thus

$$M|_C \cong \bigoplus_{\substack{0\leqslant k\leqslant n-2\\ x\in\mathbb{F}_q^*}} \mathbb{C}Cd_1(x)\sigma_k b \cong \bigoplus_{\substack{0\leqslant k\leqslant n-2\\ x\in\mathbb{F}_q^*}} \mathbb{C}Cd_n\left(x^{-1}\right)\sigma'_k b,$$

and

$$\mathbb{C}Cd_1(x)\sigma_k b \cong_{\mathbb{C}C} \mathbb{C}Cd_n\left(x^{-1}\right)\sigma'_k b,$$

as required.

9. Proof of Corollary 2.4

Since right multiplication by $\sigma_k b$ maps $\mathbb{C}Cd_1(x)e_{\sigma_k^{-1}}d_1(x)^{-1}$ to $\mathbb{C}Cd_1(x)\sigma_k b = \mathbb{C}Cd_1(x)e_{\sigma_k^{-1}}\sigma_k b$, it is clear that $\mathbb{C}Cd_1(x)\sigma_k b$ is a homomorphic image of the space $\mathbb{C}Cd_1(x)e_{\sigma_k^{-1}}d_1(x)^{-1}$ (in fact, this map is isomorphic). Hence Theorem 2.3(i) gives that the irreducible components of $\mathbb{C}Cd_1(x)\sigma_k b$ and those of $\mathbb{C}Cd_n(y)\sigma'_l b$ are pairwise non-isomorphic unless $l = k$ and $y = x^{-1}$. So by Theorem 2.3 it follows that $M|_C$ is multiplicity-free. Consequently, the isomorphism signs in the statement of Theorem 2.3 can be replaced by equalities.

Index of notation

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